Classicality and anticlassicality measures of pure and mixed quantum states

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Abstract

We introduce a simple measure of "classicality" of pure and mixed quantum states as a maximum value of the Hilbert–Schmidt "scalar products" between the renormalized statistical operators of the state concerned and all displaced thermal states. Choosing Fock states as the reference set, we introduce the measure of "anticlassicality". Both measures are illustrated for the Fock, coherent phase, and generic mixed Gaussian states. Gaussian states are shown to be the closest to thermal states possessing the same degree of quantum purity. On the contrary, Fock states appear to be more close to mixed thermal states than to pure coherent states.

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1 Introduction

Since Glauber's paper [1], quantum states for which the so-called P-distribution [2] is non-positive or more singular than delta function are called "nonclassical". For the past decades, many authors proposed different *quantitative* measures of "nonclassicality". It seems impossible to reduce all variety of quantum states in infinite-dimensional Hilbert spaces to some unique parameter. Therefore, different existing approaches should be considered sooner as complementary rather than competitive.

Historically, the first approach was based on the analysis of deviations from the Poissonian photon statistics inherent to the Klauder–Glauber–Sudarshan [1–3] *coherent states*, which are known to be the only "classical" *pure* states [4–6]. The examples are Mandel's "Q-parameter" [7] and its various generalizations [8–10]. The second direction is to evaluate the volume of that part of phase space where some quasiprobability distribution assumes negative values [11–19] (see also [20, 21]). There are also other approaches [22–24].

Here we follow the direction opened by Hillery [25]. It consists in evaluating some kinds of *distances* in the Hilbert space between the state concerned and a family of states which are assumed to be "classical". There exist many different definitions of the distance, besides the "trace distance" used in [25]. In particular, distances along geodesics on curved manifold supplied with Riemannian metrics for coherent, squeezed, displaced, and other states were studied in [26–28]. Other examples are the Monge distance [29], "polarized" and "classical-like" distances [30]. The most simple from the point of view of calculations is the Hilbert–Schmidt distance, used in [30–36],

$$\begin{aligned} d^2_{HS}(\hat{\rho},\hat{\rho}_c) &= \mathrm{Tr}(\hat{\rho}-\hat{\rho}_c)^2 \\ &= \mathrm{Tr}(\hat{\rho}^2)+\mathrm{Tr}(\hat{\rho}_c^2)-2\,\mathrm{Tr}(\hat{\rho}\hat{\rho}_c). \end{aligned} \tag{1}$$

Here $\hat{\rho}$ is the statistical operator of the quantum state concerned, and $\hat{\rho}_c$ is related to the reference "classical" state.

For *pure* quantum states, $\hat{\rho}=|\psi\rangle\langle\psi|$, the reference states are usually identified with *coherent states* $\hat{\rho}_c=|\alpha\rangle\langle\alpha|$, as soon as the latter are assumed to be "the most classical" ones [4–6]. Then the calculation of distance is reduced to calculating the scalar product $\langle\alpha|\psi\rangle$. This scalar product is the principal ingredient of many other distances, such as the Fubiny–Study distance [37] $d^2_{FS}=1-|\langle\psi_1|\psi_2\rangle|^2$ (we use here slightly different notation), or Wootters' distance [38, 39] $d_W=\cos^{-1}(|\langle\psi_2|\psi_1\rangle|)$ (see [30] for more references).

However, coherent states (whose P-distributions are delta-functions) represent only a small subset "at the border" of the set of all "classical" states. Therefore, it seems reasonable to enlarge the family of reference states $\hat{\rho}_c$, incorporating [35] all *displaced thermal states*, in complete correspondence with original paper [1].

Unfortunately, even the Hilbert–Schmidt distance becomes complicated in such a case, because the presence of three terms given in the second line of Eq. (1) does not permit to find the minimum of this expression analytically (except for the most simple cases), when one deals with *mixed* states. A possibility to simplify calculations by adjusting the "purity" $\mathrm{Tr}(\hat{\rho}_c^2)$ of the reference states to the purity $\mathrm{Tr}(\hat{\rho}^2)$ of the state concerned was discussed in [35].

The aim of our article is to consider another possibility, which permits us to find analytical expressions for most known families of nonclassical states. Namely, in the next section we introduce a new "classicality measure", which is proportional to (*but not identical with*)

the last term in (1), $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$. We analyze this measure for the Fock and generic Gaussian mixed states. In Sec. 3 we consider the "anticlassicality measure", based on using "the most nonclassical" Fock states as the reference ones. A brief discussion of results and perspectives is given in the concluding Sec. 4

2 Classicality measure

As a matter of fact, practically all information about the closeness of the states $\hat{\rho}$ and $\hat{\rho}_c$ is contained in the term $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$, whereas the presence of the term $\mathrm{Tr}(\hat{\rho}_c^2)$ in (1) only complicates the search of minimum of d_{HS}^2, adding no significant information. Therefore, it seems reasonable to consider only the term $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$, searching for its *maximum* with respect to the enlarged family of "classical" states, consisting of all displaced thermal states [40–42]

$$\hat{\rho}_c = \hat{D}(\alpha)\hat{\rho}_{th}\hat{D}^{-1}(\alpha). \tag{2}$$

Here $\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right)$ is the well-known displacement operator [1–3] written in terms of the bosonic ahhihilation and creation operators, $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{\rho}_{th}$ is the thermal state,

$$\hat{\rho}_{th} = (1-\eta)\sum_{k=0}^{\infty}\eta^k|k\rangle\langle k|, \quad 0 \le \eta < 1. \tag{3}$$

If we confine ourselves to the special case of $\eta = 0$, when $\rho_c = |\alpha\rangle\langle\alpha|$, and consider only pure quantum states $\hat{\rho} = |\psi\rangle\langle\psi|$, then the quantity $\max_\alpha |\langle\alpha|\psi\rangle|^2$ seems a good measure of "classicality", because it equals 1 for coherent states and is less than 1 for all other pure states. However, dealing with the generic case of mixed states, one meets certain difficulties: on the one hand, the measure of classicality for any thermal state must be the same as for coherent states, i.e., it must be equal to 1 according to the normalization chosen, but on the other hand, $\mathrm{Tr}(\hat{\rho}_c^2) < 1$. Therefore, one has to find a suitable generalization of the scalar product between pure states to the case of mixed states, which would result in the unit value for identical states.

One possible solution, satisfying many additional requirements, was given by Uhlmann [43], who showed that the generalization of the quantity $|\langle\psi_1|\psi_2\rangle|$ can be taken in the form $\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}$. However, the calculation of this trace is rather complicated problem, although certain progress has been achieved recently for Gaussian states (in the case of infinite-dimensional Hilbert space) [44–47].

We prefere to follow a more straightforward way: simply to replace the operators $\hat{\rho}$ and $\hat{\rho}_c$ in $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$ by the *renormalized* operators $\hat{\rho}'$ and $\hat{\rho}_c'$, where

$$\hat{\rho}' \equiv \hat{\rho}/\sqrt{\mathrm{Tr}\,(\hat{\rho}^2)}\,. \tag{4}$$

Therefore, we define the "classicality measure" as

$$\mathcal{C} = \max_{\rho_c}\mathrm{Tr}\,(\hat{\rho}'\hat{\rho}_c')\,. \tag{5}$$

For pure states, $\hat{\rho} = |\psi\rangle\langle\psi|$, we have

$$\mathcal{C} = \max_{\rho_c}\langle\psi|\hat{\rho}_c'|\psi\rangle. \tag{6}$$

In modern literature, the "generalized scalar products", such as $\mathrm{Tr}\sqrt{\hat{\rho}_1^{1/2}\hat{\rho}_2\hat{\rho}_1^{1/2}}$ or $\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2)$, are frequently called [48, 49] "fidelities" ("Bures–Uhlmann" or "Hilbert–Schmidt", respectively). In particular, a possibility of using the "modified fidelity" $\mathrm{Tr}(\hat{\rho}_1'\hat{\rho}_2')$ was mentioned in [50]. Note that if only one of the states is pure, then the "modified fidelity" does not coincide with $\langle\psi_1|\hat{\rho}_2|\psi_1\rangle$, but differs from it by the factor $[\mathrm{Tr}(\hat{\rho}_2^2)]^{-1/2}$. Sometimes [48,49] this is considered as a flaw. But in our problem the factor discussed has real physical meaning, because it permits us to distinguish mixed states from their pure "partners" possessing the same probability distribution functions of quanta (see examples below). Another difference between the cited papers and our approach is that we consider "fidelity" not to a fixed state, but to the whole family of "classical" states.

We shall use the notation $f(\eta,\alpha)$ for the function $\mathrm{Tr}(\hat{\rho}'\hat{\rho}_c')$, whose maximum gives the value of classicality. Sometimes, it can be difficult to find a maximum with respect to both variables. Then one can calculate partial maxima with respect to one variable, fixing zero value for another. We shall denote such "reduced classicalities" as $\widetilde{\mathcal{C}}_\alpha$ or $\widetilde{\mathcal{C}}_\eta$, where the subscript indicates the parameter, with respect to which the maximum was calculated.

2.1 Classicality of the Fock states

For the Fock state $|n\rangle$, the quantity $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$ is reduced to the probability $\langle n|\hat{\rho}_c|n\rangle$ of discovering n quanta in the displaced thermal state. This probability was calculated by many authors, e.g., in [40–42, 51]. Using the results of [51] and taking into account the expression for purity of the thermal state (it does not depend on the shift parameter α),

$$\mathrm{Tr}\left(\hat{\rho}_c^2\right)=\frac{1-\eta}{1+\eta}, \tag{7}$$

we arrive at the function

$$\begin{aligned} f_n(\eta,\alpha) &= \sqrt{1-\eta^2}\,\eta^n\exp\left(-|\alpha|^2\sqrt{1-\eta^2}\right) \\ &\quad\times L_n\left(-|\alpha|^2(1-\eta)^2/\eta\right), \end{aligned} \tag{8}$$

where $L_n(z)$ is the Laguerre polynomial. For $\eta=0$ we have the well-known Poissonian distribution of the coherent state,

$$f_n(0,\alpha)=\exp\left(-|\alpha|^2\right)|\alpha|^{2n}/n!,$$

which has maximum at $|\alpha|^2=n$. Therefore, confining ourselves to pure coherent reference states, we obtain for the reduced classicality of the Fock state the formula (cf. [35]) $\widetilde{\mathcal{C}}_\alpha^{(n)}=e^{-n}n^n/n!$, which results in $\widetilde{\mathcal{C}}_\alpha^{(n)}\approx(2\pi n)^{-1/2}$ for $n\gg 1$.

On the other hand, the function $f_n(\eta,0)$ has maximum at $\eta_n=\sqrt{n/(n+1)}$, so that

$$\widetilde{\mathcal{C}}_\eta^{(n)}=\sqrt{n^n/(1+n)^{n+1}}\,. \tag{9}$$

One can verify that $\widetilde{\mathcal{C}}_\eta^{(n)}>\widetilde{\mathcal{C}}_\alpha^{(n)}$ for any $n\geq 1$. In particular, $\widetilde{\mathcal{C}}_\eta^{(1)}=1/2$ and $\widetilde{\mathcal{C}}_\alpha^{(1)}=1/e$, whereas for $n\gg 1$ we have $\widetilde{\mathcal{C}}_\eta^{(n)}\approx(en)^{-1/2}$. Moreover, the function $f_n(\eta,\alpha)$ has *negative*

derivative with respect to $|\alpha|^2$ at the point $|\alpha|^2 = 0$ for any η_n. Therefore, the maxima of $f_n(\eta, \alpha)$ at the points $(\eta_n, 0)$ are global, whereas the maxima at the points $(0, n)$ are only local: see Fig. 1. This means that the classicality of the Fock state is given by (9), and that the pure state $|n\rangle$ is more close not to the pure (displaced) coherent state with $\overline{n}_{coh} \equiv |\alpha|^2 = n$, but to the mixed (undisplaced) thermal state with

$$\overline{n}_{th} \equiv \eta/(1-\eta) = \sqrt{n}\left(\sqrt{n} + \sqrt{n+1}\right)$$

(i.e., $\overline{n}_{th} \approx 2n$ for $n \gg 1$) and

$$\mathrm{Tr}\left(\hat{\rho}_c^2\right) = \left(\sqrt{n} + \sqrt{n+1}\right)^{-2}.$$

2.2 Classicality of Gaussian states

Generic Gaussian states are characterized [51] by two variances of the quadrature components, σ_q and σ_p, their covariance, σ_{pq}, and two displacement parameters in the phase plane qp. It is clear, however, that for the fixed (co)variances, the quantity $\mathrm{Tr}(\hat{\rho}\hat{\rho}_c)$ is maximal for coinciding displacement parameters of $\hat{\rho}$ and $\hat{\rho}_c$, which means that it is sufficient to consider the states with zero displacements, i.e., to find the maximum of the function of single variable $F(\eta) \equiv f(\eta, 0)$. Since the statistical operator of the thermal state is diagonal in the Fock basis, the function $F(\eta)$ can be easily expressed in terms of the *generating function*

$$G(z) \equiv \sum_{n=0}^{\infty} p_n z^n$$

for the photon distribution $p_n \equiv \langle n|\hat{\rho}|n\rangle$:

$$F(\eta) = \mu^{-1/2}\sqrt{1-\eta^2}\, G(\eta), \quad \mu \equiv \mathrm{Tr}(\hat{\rho}^2). \tag{10}$$

The function $G(z)$ for generic Gaussian states was calculated by means of different methods, e.g., in [41, 51, 52]. Using the form given in [51], we arrive at the function

$$F(\eta) = 2\left[\frac{\mu(1-\eta^2)}{a - 2b\eta + c\eta^2}\right]^{1/2}, \tag{11}$$

where

$$a = 1 + \mu^2 + 2T\mu^2, \quad b = 1 - \mu^2, \quad c = 1 + \mu^2 - 2T\mu^2,$$
$$\mu^2 = \left[4\left(\sigma_p\sigma_q - \sigma_{pq}^2\right)\right]^{-1}, \quad T = \sigma_p + \sigma_q\,.$$

One can easily verify that the maximum of function (11) is attained for $\eta_* = (1-\mu)/(1+\mu)$, which means that the thermal state which is most close to the given Gaussian state has the same purity: $\mu_c = \mu$. The classicality of the Gaussian states is given by a simple formula

$$\mathcal{C}^{(G)} = \sqrt{\frac{2}{1+\mu T}}\,. \tag{12}$$

Note that $\mu T \geq 1$ for Gaussian states, and the equality holds for thermal states. The parameter T is related to the mean number of quanta $\overline{n}$ in the unshifted Gaussian state as $T = 1 + 2\overline{n}$. Therefore, the classicality of *pure* unshifted Gaussian states (which are nothing but *squeezed vacuum states*) equals $\mathcal{C}^{(squv)} = (\overline{n}+1)^{-1/2}$. If $\mu < 1$, then the minimal mean number of quanta in the Gaussian state equals $\overline{n}_{min} = (1-\mu)/(2\mu)$. However, mixed Gaussian states remain *unsqueezed* (that means, no one quadrature component can have the variance less than the vacuum state value 1/2 in dimensionless units), if $\overline{n} < \overline{n}_c = (1-\mu^2)/(2\mu^2)$. Such states possess positive P-distributions [40], and it was proposed in [47] to extend the family of "classical" states, including all mixed unsqueezed Gaussian states. We suppose to study such an "extended classicality" in another paper.

There exist *pure* states possessing the same photon distribution function as thermal ones; usually they are called *coherent phase states* [53, 54]:

$$|\varepsilon\rangle = \sqrt{1-|\varepsilon|^2}\sum_{n=0}^{\infty}\varepsilon^n\,|n\rangle. \tag{13}$$

Although the state (13) is most close (at least with respect to the reduced $\widetilde{\mathcal{C}}_\eta$-measure) to the thermal state with $\eta = |\varepsilon|^2$, its η-classicality is less than 1 (precisely because we use renormalized operators in the definition (5), the function $F(\eta)$ in Eq. (10) contains the factor $\mu^{-1/2}$, but this factor equals unity now, as soon as $\hat{\rho}'_\varepsilon = \hat{\rho}_\varepsilon$):

$$\widetilde{\mathcal{C}}_\eta^{(\varepsilon)} = \sqrt{\frac{1-|\varepsilon|^2}{1+|\varepsilon|^2}} = (1+2\overline{n})^{-1/2}\ ,$$

in accordance with our feeling that pure states, in general, are less classical than their mixed partners. For highly excited states with the same mean value $\overline{n} \gg 1$ we obtain the asymptotical relations

$$\mathcal{C}^{(squv)} \approx \sqrt{2}\,\widetilde{\mathcal{C}}_\eta^{(\varepsilon)} \approx \sqrt{e}\,\mathcal{C}^{(Fock)}.$$

3 Anticlassicality measures

As soon as the Fock states are usually considered as "the most quantum" states, we can use them as reference states for the definition of the degree of "anticlassicality":

$$\mathcal{A} = \max_n \langle n|\hat{\rho}|n\rangle. \tag{14}$$

Perhaps, it could be worth including the displaced Fock states, $|\tilde{n}\rangle \equiv \hat{D}(\alpha)|n\rangle$, into the set of reference states, but here we shall not do it. Actually, we can define two measures: one given by (14), where integer n runs over all integers, *including* $n = 0$, and another (denoted as $\mathcal{A}_1$), where the vacuum state $|0\rangle$ (which is distinguished from all other states) is excluded from the set of reference states. To see the difference, let us consider an example of coherent states $|\alpha\rangle$. One can easily find that the dependence of $\mathcal{A}$ on the mean photon number $\overline{n} = |\alpha|^2$ has different analytical forms in the intervals $k \leq \overline{n} \leq k+1$, $k = 0, 1, \ldots$:

$$\mathcal{A}^{(\alpha)}(\overline{n}) = \exp(-\overline{n})\,\overline{n}^k/k!\,, \quad k \leq \overline{n} \leq k+1. \tag{15}$$

The function given by (15) is continuous, but it has jumps of derivatives at the points $\overline{n}_k = k$: the right derivatives at these points are equal to zero, while the left derivatives have finite negative values. However, $\mathcal{A}^{(\alpha)}(\overline{n})$ goes to 1 as $\overline{n} \to 0$, and this may seem strange.

The function $\mathcal{A}_1^{(\alpha)}(\overline{n})$ coincides with $\mathcal{A}^{(\alpha)}(\overline{n})$ for $\overline{n} \geq 1$. But for $\overline{n} < 1$ its behaviour is quite different:

$$\mathcal{A}_1^{(\alpha)}(\overline{n}) = \overline{n} \exp\left(-\overline{n}\right), \quad 0 \leq \overline{n} \leq 1.$$

The dependence $\mathcal{A}_1^{(\alpha)}(\overline{n})$ is shown in Fig. 2. Asymptotically, for $\overline{n} \gg 1$, we have (using the Stirling formula) $\mathcal{A}_1^{(\alpha)} \approx (2\pi\overline{n})^{-1/2}$.

As another example we consider the squeezed vacuum state. It is known that the photon distribution function p_n of this state is different from zero only for even values $n = 2m$, and it can be expressed in terms of the mean number of photons as [51]

$$p_{2m}^{(sqv)} = (1+\overline{n})^{-1/2} \frac{(2m)!}{(2^m\, m!)^2} \left(\frac{\overline{n}}{1+\overline{n}}\right)^m. \tag{16}$$

The sequence p_{2m} decreases monotonously. Therefore we obtain a monotonous dependence

$$\mathcal{A}^{(sqv)}(\overline{n}) = (1+\overline{n})^{-1/2},$$

whereas the function

$$\mathcal{A}_1^{(sqv)}(\overline{n}) = \frac{\overline{n}}{2(1+\overline{n})^{3/2}}$$

has maximum $\mathcal{A}_{1max}^{(sqv)} = 1/(3\sqrt{3})$ for $\overline{n} = 2$ (and the most close Fock state is $|2\rangle$).

For the coherent phase state (13) we obtain

$$\mathcal{A}^{(\varepsilon)}(\overline{n}) = (1+\overline{n})^{-1}, \quad \mathcal{A}_1^{(\varepsilon)}(\overline{n}) = \overline{n}(1+\overline{n})^{-2},$$

with $\mathcal{A}_{1max}^{(\varepsilon)} = 1/4$ at $\overline{n} = 1$. According to definition (14), the same results must hold for thermal states, since they have the same photon distribution function $\langle n|\hat{\rho}|n\rangle$. However, we can distinguish pure and mixed states having the same diagonal elements in the Fock basis, playing again with the purity parameter $\mu = \mathrm{Tr}(\hat{\rho}^2)$. But in contrast to the case of classicality parameter, now we should, sooner, *multiply* the quantity $\langle n|\hat{\rho}|n\rangle$ by μ (or some power of μ, if one wishes), rather than divide it by μ (or $\sqrt{\mu}$), because the intuition tells us that mixed states are more anticlassical than pure ones. Adopting the definitions

$$\widetilde{\mathcal{A}} = \mu \max_n \langle n|\hat{\rho}|n\rangle, \quad \widetilde{\mathcal{A}}_1 = \mu \max_{n\geq 1} \langle n|\hat{\rho}|n\rangle, \tag{17}$$

we obtain for the thermal states the expressions

$$\widetilde{\mathcal{A}}^{(th)}(\overline{n}) = \left[(1+\overline{n})(1+2\overline{n})\right]^{-1},$$

$$\widetilde{\mathcal{A}}_1^{(th)}(\overline{n}) = \frac{\overline{n}}{(1+\overline{n})^2(1+2\overline{n})}.$$

These three cases are illustrated in Fig. 3. The asymptotical dependences at $\overline{n} \gg 1$ are as follows:

$$\mathcal{A}_1^{(sqv)} \sim \frac{1}{2\sqrt{\overline{n}}}, \quad \mathcal{A}_1^{(\varepsilon)} \sim \frac{1}{2\overline{n}}, \quad \mathcal{A}_1^{(th)} \sim \frac{1}{2\overline{n}^2}.$$

For generic (mixed and shifted) Gaussian states, the dependences $\mathcal{A}_1(\overline{n})$ and $\widetilde{\mathcal{A}}_1(\overline{n})$ may be more complicated, especially in the regime of strong irregular oscillations of the photon distribution function p_n [55, 56]. We reserve this case for another study.

4 Conclusion

We have introduced new measures of "classicality" and "anticlassicality" of quantum states, which characterize the "fidelities" of the chosen state to the whole families of states considered as "classical" or "maximally quantum". Due to simplicity, these measures can be calculated analytically for many important sets of quantum states, thus providing additional information on their properties. In particular, we have shown that the most close to arbitrary Gaussian states are thermal states with the same degree of quantum mixing (purity). At the same time, there are (mixed) thermal states which are more close to the Fock states than (pure) coherent states. As soon as the new measures distinguish coherent quantum superpositions ("cat states") from quantum mixtures, they could be used in studies of decoherence processes. Also, it would be interesting to establish relations between the new and existing measures of classicality, such as Mandel's parameter, "nonclassical depth" [13], and so on, or to try to "order" different families of quantum states in accordance with their degrees of (anti)classicality. We hope to report on the results of studies in these directions somewhere.

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Figure 1: The renormalized Hilbert–Schmidt fidelity between the Fock state $|3\rangle$ and displaced thermal states.

Figure 2: The anticlassicality measure $\mathcal{A}_1^{(\alpha)}$ of coherent states versus the mean number of quanta $\overline{n}$.

Figure 3: The anticlassicality measures $\mathcal{A}_1(\overline{n})$ for the squeezed vacuum and phase coherent states, compared with the modified measure $\widetilde{\mathcal{A}}_1(\overline{n})$ for the thermal states.